COMMENTS ON: THE ALGEBRA AND GEOMETRY OF STEINER AND OTHER QUADRATICALLY PARAMETRIZABLE SURFACES

ADAM COFFMAN

Last update: Jan. 2, 2019.

1. Errata

The following typos appear in the published paper, [CSS].

- (p. 270) The derivative, equation (34), should read

$$\frac{\partial f_3}{\partial x} = [x_2(x_3^2 + x_4^2) - 2x_1x_4^2, x_1(x_3^2 + x_4^2),$$
$$2x_1x_2x_3 - 4x_3(x_3^2 + x_4^2), 2x_1x_2x_4 - 2x_1^2x_4 - 4(x_3^2 + x_4^2)x_4].$$

  This corrects the lower right 2×2 block of the Hessian, equation (35), to

$$\left.\begin{matrix} 2x_1x_2 - 12x_3^2 - 4x_4^2 & -8x_3x_4 \\ -8x_3x_4 & 2x_1x_2 - 2x_1^2 - 12x_4^2 - 4x_3^2 \end{matrix}\right] .$$

- (p. 273) Below equation (58) should read "As in the case of Σ_7 and Σ_8"
- (p. 279) **Theorem 7.** should read "*The order of* Σ *is* $4 - \nu_\Sigma$."
- (p. 281) Above equation (105) should read "Since $\mathcal{P}(\Sigma) \not\subseteq \mathbb{M}_2$, $e \neq 0$."
- (p. 284) Step (2) in Section **6.** should read "$\det\{\lambda\mathbf{M} + \mu\mathbf{N}\}$."

2. Updates

The references mention unpublished notes of A. Schwartz and C. Stanton — a version from 1988 is available from me, on request. Another pre-1996 source on this topic, with some details on the matrix calculations leading to the classification theorem, is [C_1], which I made available online in 2018.

It should have been mentioned that Coffman's research was supported in part by a National Science Foundation Research Experience for Undergraduates program in the summer of 1990.

Since publication, the following related article has come to our attention: [D]. The topic of projections of the real Veronese variety has more recently been considered in [C_3]. Also see my web page on Steiner surfaces, currently at this address: [C_2].

3. Citations

Our article is cited in these academic papers: [A_1], [A_2], [A_3], [ABB], [AMT], [AS_1], [AS_2], [BJKL], [BCF], [BEG], [CFRV], [EGL_1], [EGL_2], [GS], [G], [HJS], [H], [HW_1], [HK], [HW_2], [HW_3], [LG], [PA], [PO], [POS], [PR], [PT], [P], [RJ], [S], [SPS], [WG], [WC], [WCD], [Zanella], [Z_1], [Z_2], [Z_3], as well as these books: [F], [KI], and this computer technical manual: [T].

References

[A_1] G. Albrecht, *Determination and classification of triangular quadric patches*, Computer Aided Geometric Design (7) **15** (1998), 675–697. MR 1632733 (99e:65190).

[A_2] G. Albrecht, *The Veronese surface revisited*, J. Geom. (1–2) **73** (2002), 22–38. MR 1916463 (2003e:65019).

[A_3] G. Albrecht, *Geometric invariants of parametric triangular quadric patches*, Int. Electron. J. Geom. (2) **4** (2011), 63–84. MR 2854276.

[ABB] F. Aries, E. Briand, and C. Bruchou, *Some covariants related to Steiner surfaces*, in Geometric Modeling and Algebraic Geometry, 31–46, Springer, Berlin, 2008. MR 2381602 (2009b:14111), Zbl 1132.14336.

[AMT] F. Aries, B. Mourrain, and J.-P. Técourt, *Quadratically parametrized surfaces: algorithms and applications*, in Geometric Modeling and Computing: Seattle 2003, M. L. Lucian and M. Neamtu (eds.), Nashboro Press, Brentwood, TN, 2004, 21–40. MR 2122113, Zbl 1060.65528.

[AS_1] F. Aries and R. Senoussi, *Approximation de surfaces paramétriques par des carreaux rationnels du second degré en lancer de rayons*, Revue de CFAO et d'informatique graphique (6) **12** (1997), 627–645.

[AS_2] F. Aries and R. Senoussi, *An implicitization algorithm for rational surfaces with no base points*, J. Symbolic Comput. (4) **31** (2001), 357–365. MR 1823071 (2002a:14063), Zbl 0981.14030.

[BJKL] B. Bastl, B. Jüttler, J. Kosinka, M. Lávička, *Computing exact rational offsets of quadratic triangular Bezier surface patches*, Computer Aided Design (2) **40** (2008), 197–209.

[BCF] R. Binotto, S. Costa, and M. C. Romero Fuster, *The curvature Veronese of a 3-manifold immersed in Euclidean space*, in Real and Complex Singularities, XIII International Workshop, São Carlos, 2014, Contemporary Mathematics **675**, 2016. MR 3578716.

[BEG] L. Busé, M. Elkadi, and A. Galligo, *Intersection and self-intersection of surfaces by means of Bezoutian matrices*, Computer Aided Geometric Design (2) **25** (2008), 53–68. MR 2388405 (2009b:65034).

[CFRV] A. Cantón, L. Fernández-Jambrina, E. Rosado María, and M. J. Vázquez-Gallo, *Implicit equations of non-degenerate rational Bezier quadric triangles*, in Curves and Surfaces, 8^{th} International Conference, Paris, 2014, Springer LNCS **9213**, 2015. MR 3440735.

[$CFRV_2$] A. Cantón, L. Fernández-Jambrina, E. Rosado María, and M. J. Vázquez-Gallo, *Geometric elements and classification of quadrics in rational Bézier form*, Journal of Computational and Applied Mathematics **300** (2016), 400–419. MR 3460308.

[C_1] A. Coffman, *A Classification of Quadratically Parametrized Maps of the Real Projective Plane*, B.S. Honors Thesis, University of Michigan, Ann Arbor, 1991. `http://users.pfw.edu/CoffmanA/`

[C_2] A. Coffman, *Steiner Surfaces* web site. `http://users.pfw.edu/CoffmanA/steinersurface.html`

[C_3] A. Coffman, *Real congruence of complex matrix pencils and complex projections of real Veronese varieties*, Linear Algebra and its Applications **370** (2003), 41–83. MR 1994320 (2004f:14026), Zbl 1049.14042.

[CSS] A. Coffman, A. J. Schwartz, and C. M. Stanton, *The algebra and geometry of Steiner and other quadratically parametrizable surfaces*, Computer Aided Geometric Design (3) **13** (1996), 257–286. MR 1390390 (97g:53012), Zbl 0875.68860.

[D] W. L. F. Degen, *The types of triangular Bézier surfaces*, in The Mathematics of Surfaces, VI (Brunel Univ., 1994), 153–170, Inst. Math. Appl. Conf. Ser. New Ser., **58**, G. Mullineux, ed., Oxford, 1996. MR 1430584 (98h:51033).

[EGL_1] M. Elkadi, A. Galligo, and T. H. Lê, *Parametrized surfaces in $\mathbb{P}^3$ of bidegree $(1,2)$*, Proceedings of ISSAC (Santander, Spain), 141–148, ACM, New York, 2004. MR 2126936 (2005j:68135).

[EGL_2] M. Elkadi, A. Galligo, and T. H. Lê, *On parametric surfaces of low degree in $\mathbb{P}^3(\mathbb{C})$*, in Algebraic Geometry and Geometric Modeling, 151–168, Springer series on Mathematics and Visualization, Berlin, 2006. MR 2279849 (2007k:14125).

[F] R. Farouki, *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable.* Geometry and Computing **1**. Springer, Berlin, 2008. MR 2365013 (2008k:65027).

[GS] A. Galligo and M. Stillman, *On the geometry of parametrized bicubic surfaces*, Journal of Symbolic Computation (1–2) **42** (2007), 136–158. MR 2284289 (2008e:14054).

[G] G. H. Georgiev, *Rational generalized offsets of rational surfaces*, Mathematical Problems in Engineering **2012** Article ID 618148, (2012), 15 pages. MR 2878968.

[HJS] M. Hauer, B. Jüttler, and J. Schicho, *Projective and affine symmetries and equivalences of rational and polynomial surfaces*, Journal of Computational and Applied Mathematics **349** (2019), 424–437. MR 3886725.

[H] D. Henrion, *Semidefinite representation of convex hulls of rational varieties*, Acta Appl. Math. (3) **115** (2011), 319–327. MR 2823121 (2012f:14112).

[HW_1] J. W. Hoffman and H. Wang, *A note on quadratically parametrized surfaces*, Algebra Colloq. (3) **21** (2014), 461–476. MR 3224670.

[HK] I. Hoveijn and O. N. Kirillov, *Singularities on the boundary of the stability domain near* 1 : 1*-resonance*, J. Differential Equations (10) **248** (2010), 2585–2607. MR 2600969 (2011h:34108).

[HW_2] Y. Huang and D. Wang, *Computing self-intersection loci of parametrized surfaces using regular systems and Gröbner bases*, SYNASC 2009 — 11^{th} International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, art. no. 5460872, 28–36.

[HW_3] Y. Huang and D. Wang, *Computing intersection and self-intersection loci of parametrized surfaces using regular systems and Gröbner bases*, Computer Aided Geometric Design (9) **28** (2011), 566–581. MR 2861556 (2012j:65046).

[KI] S. Krivoshapko and V. N. Ivanov, *Encyclopedia of Analytical Surfaces*, Springer, 2015. MR 3309742.

[LG] T.-H. Lê and A. Galligo, *General classification of* $(1,2)$ *parametric surfaces in* $\mathbb{P}^3$, in Geometric Modeling and Algebraic Geometry, 93–113, Springer, Berlin, 2008. MR 2381606 (2009a:14075).

[PA] R. Paul and S. Anand, *A new Steiner patch based file format for Additive Manufacturing processes*, Computer-Aided Design **63** (2015), 86–100.

[PO] M. Peternell and B. Odehnal, *Convolution surfaces of quadratic triangular Bézier surfaces*, Computer Aided Geometric Design (2) **25** (2008), 116–129. MR 2388409 (2008m:65052).

[POS] M. Peternell, B. Odehnal, and M. Sampoli, *On quadratic two-parameter families of spheres and their envelopes*, Computer Aided Geometric Design (4–5) **25** (2008), 342–355. MR 2408097 (2009b:65046).

[PR] J. Peters and U. Reif, *The 42 equivalence classes of quadratic surfaces in affine* n*-space*, Computer Aided Geometric Design (5) **15** (1998), 459–473. MR 1624047 (99e:14063).

[PT] V. Petrushchenko and V. Timorin, *On maps taking lines to plane curves*, Arnold Mathematical Journal (1) **2** (2016), 1–20. MR 3460040.

[P] R. Piene, *Singularities of some projective rational surfaces*, in Computational Methods for Algebraic Spline Surfaces, 171–182, Springer, Berlin, 2005. MR 2127188 (2005m:14060).

[RJ] K. Rittenschober and B. Jüttler, *Rational mappings associated with space filling line congruences*, in Geometric Modeling and Computing: Seattle 2003, M. L. Lucian and M. Neamtu (eds.), Nashboro Press, Brentwood, TN, 2004, 481–490. MR 2122140, Zbl 1060.65547.

[S] M. Saniga, *Cremona transformations and the conundrum of dimensionality and signature of macro-spacetime*, Chaos, Solitons and Fractals (12) **12** (2001), 2127–2142. MR 1859514 (2003a:51019).

[SPS] J. R. Sendra, M. Peternell, and J. Sendra, *Cissoid constructions of augmented rational ruled surfaces*, Computer Aided Geometric Design **60** (2018), 1–9. MR 3765807.

[T] M. Trott, *The Mathematica GuideBook for Graphics*, Springer, 2004.

[WG] H. Wang and R. Goldman, *Syzygies for translational surfaces*, J. Symbolic Computation **89** (2018), 73–93. MR 3804807.

[WC] X. Wang and F. Chen, *Implicitization, parameterization and singularity computation of Steiner surfaces using moving surfaces*, J. Symbolic Computation (6) **47** (2012), 733–750. MR 2908591.

[WCD] X. Wang, F. Chen, and J. Deng, *Implicitization and parametrization of quadratic surfaces with one simple base point*, Proceedings of ISSAC (Linz/Hagenberg, Austria), 31–38, 2008. MR 2500370 (2010h:68232).

[Zanella] C. Zanella, *On finite Steiner surfaces*, Discrete Mathematics (3) **312** (2012), 652–656. MR 2854812.

[Z_1] S. Zubė, *Bidegree* $(2,1)$ *parametrizable surfaces in projective* 3*-space*, Liet. Mat. Rink. (3) **38** (1998), 379–402; translation in Lithuanian Math. J. (3) **38** (1999), 291–308. MR 1657885 (99m:51030).

[Z_2] S. Zubė, *Real ruled degree four toric surfaces in projective* 3*-space*, Proceedings of 39^{th} Conference of Lithuanian Mathematical Society, 191–195, 1998.

[Z_3] S. Zubė, *Correspondence and* $(2,1)$*-Bézier surfaces*, Liet. Mat. Rink. (1) **43** (2003), 99–122; translation in Lithuanian Math. J. (1) **43** (2003), 83–102. MR 1996756 (2004k:14061).

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499

E-mail address: `CoffmanA@pfw.edu`

Our construction relies on projective geometry and is grounded on the pencil of quadrics circumscribed to a tetrahedron formed by vertices of the control net and an additional point which is required for the Steiner surface to be a non-degenerate quadric. Read full text view PDF. Post comment. Comments. There are no comments yet. POST REPLY. Ã—.Â [4] Coffman, A., Schwartz, A.J., Stanton, C.: The algebra and geometry of steiner and other quadratically parametrizable surfaces. Computer Aided Geometric Design 13(3), 257 â€“ 286 (1996). [5] Degen, W.: The types of triangular BÃ©zier surfaces. The algebra and geometry of Steiner and other quadratically parametrizable surfaces. Computer Aided Geometric De-sign, 13:257â€“286.zbMATHCrossRefMathSciNetGoogle Scholar. 8. Degen, W. (1996). The types of triangular BÃ©zier surfaces. In Mullineux, G., editor, The Mathematics of Surfaces IV, volume 38 of The Institute of Mathematics and its Applica-tions Conference, pages 153â€“171.Â A Treatise on the Analytic Geometry of Three Dimensions. Longman, Greens and co., fifth edition. Revised by Reginald A. P. Rogers.Google Scholar. 15. Sederberg, T. W. and Chen, F. (1995). Implicitization using moving curves and surfaces. Computer Graphics Siggraph 1995, 18:301â€“308.Google Scholar. Copyright information. Quadratically parameterizable rational surfaces are classied in [3]. A well-known example is Steinerâ€™s Roman surface, a non-orientable quartic surface with three double lines, which is parameterized as follows: V = {v âˆˆ R3. : v1(x).Â Lectures on modern convex optimization. SIAM, 2001. [2] E. Brieskorn, H. KnÂ¨orrer.Â The algebra and geometry of Steiner and other quadratically parametrizable surfaces. Computer Aided Geometric Design, 13:257-286, 1996. [4] J. Harris. Algebraic geometry: a rst course. Springer, 1992. [5] J. W. Helton, J. Nie. [9] Coï¬€man, A., Schwartz, A.J. and Stanton, Ch., The algebra and geometry of Steiner and other quadratically parametrizable surfaces, Comput. Aided Geom. Design 13, 257â€“286 (1996). [10] Degen, W.L.F., The types of triangular BÃ©zier surfaces, in The Mathematics of Surfaces VI (G. Mullineux, ed.), The IMA Conference Series No. 58, Clarendon Press Oxford, 153â€“170 (1996).Â [23] Vaisman, I., Analytical Geometry, Series on University Mathematics Volume 8, World Scientiï¬c (1997). [24] Varady, T., Survey and new results in nâ€“sided patch generation, in The Mathematics of Surfaces II, (R. Martin, ed.), Oxford Univ. Press, Oxford, 203â€“235 (1987). AyrÄ±ntÄ±lar.