

Higher-Dimensional Category Theory

The architecture of mathematics

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The history of maths shows that its greatest contribution to science, culture and technology has been in terms of expressive power, to give a language for intuitions which enables exact description, calculation, deduction.

—RONALD BROWN

to 'Categories' e-mailing list, 21st June 2000

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Foreword

Most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity.

—G. H. HARDY

A Mathematician's Apology

It would be absurd to suggest that the learning of German grammar had much to do with research on, say, the work of Schopenhauer. However, analogous such assumptions abound regarding mathematics, if only because it is so hard to discuss mathematical research without using obscure technical language.

For this reason, I will frequently attempt to give, not a precise account of my research, but an impressionistic idea of it. Impressionism may not stand up to rigorous analysis, but it carries with it a different sort of clarity. Its apparently hazy interpretations of subject material evoke rather than represent; evocative images reach a wider audience more directly than accounts requiring a specialist interpreter to mediate.

In place of hazy brushstrokes, I will use analogies to create those evocative images. An analogy is, by definition, not an actual description, so is bound to be limited in scope. However, I hope that the analogies will help to give a sense of what my research is, by comparing the very abstract notions to much more universally appealing ones.

Introduction

Before constructing a new building, the site must first be cleared of pre-existing material, and then the foundations must be laid. Only then can the building itself begin to emerge.

Category Theory is the mathematics of mathematics.

In order to understand this bold statement it is evidently necessary (twice over) to understand what mathematics *is*. By this ‘understanding’ I do not mean a working knowledge of advanced mathematics. I have no intention of providing a ‘crash course’ in advanced algebra in order to describe my research; this would be no more interesting to the reader than a dictionary preceding a paper in Esperanto.

Category Theory is the study and formalisation of the way mathematics ‘works’, or the *essence* of mathematics, so it is this essence that I will first try to describe. I will then explain how Category Theory studies this essence.

Then as a final preliminary, I will introduce the idea of *dimensions* in category theory. I stress that these are not physical dimensions, but rather, conceptual ones, or layers of complication.

Only when all these foundations are in place will it be possible for me to start describing my research. The field of Higher-Dimensional Category Theory is young within Category Theory, since of course, the ‘low’ dimensions

had to be examined first. The ‘high’ dimensions are not fully understood; various attempts have been made, but the relationship between them has been unclear. Broadly speaking, my work has focused on the huge task of unifying the different theories.

However, even before doing any of the above, I must clear the psychological site of possibly counterproductive notions of mathematics, that is, explain what mathematics *is not*. Mathematics is not ‘the study of numbers’. I urge the reader not to be distracted by thoughts of arduous maths lessons at school, lengthy calculations culminating in the wrong answer, or endless memory-defying formulae. This is merely the grammar; it is not necessary to understand grammatical intricacies to appreciate poetry.

Chapter 1

Foundations

1.1 Theory: What is mathematics?

A mathematician, like a painter or a poet, is a maker of forms.

—G. H. HARDY

A Mathematician's Apology

Mathematics is the rigorous study of conceptual systems. It may be seen as having two general roles:

1. To provide a language for making precise statements about concepts, and a system for making clear arguments about them.
2. To idealise concepts so that a diverse range of notions may be compared and studied simultaneously by focusing only on relevant features common to all of them.

Mathematics as a language has developed with a general aim of eliminating ambiguity. What has been sacrificed in pursuit of this ideal?

The most obvious sacrifice is that of *scope*. Rigour cannot be imposed upon every element of human consciousness. (Indeed, it may be precisely this impossibility that makes the human consciousness so endlessly rich.) In order to maintain rigour, we must be carefully precise about the issues we are considering, and the context in which we are considering them.

A conceptual system is a system involving only ideas rather than physical phenomena. Physical systems pre-existing in the physical world around us already have properties which we can only observe and therefore not control. Scientific experiments seek to isolate parts of physical systems in order better to study their properties; a *conceptual* system might be seen as the purest form of such isolation. It is not only objects that are isolated, but *characteristics* of those objects.

Generally, we study such a system by defining the components we desire as our ‘building blocks’, together with any rules we require them to satisfy.

Examples

1. Building blocks: a/b where a and b are whole numbers and b is not 0
Rules: $a/b = c/d$ if $ad = bc$
System: fractions (rational numbers)
2. Building blocks: whole numbers
Rules: $1 + 1 = 0$
System: Binary numbers
3. Building blocks: bricks
Rules: bricks may be placed on top of one another or end to end
System: brick walls

4. Building blocks: people

Rules: the College rules

System: Gonville and Caius College

Whether we have created or discovered a system is a moot philosophical point. Inasmuch as mathematics is the *study* of conceptual systems, we have ‘created’ a way of describing and thereby studying such a system, even though, perhaps, we have not created the system itself.

A small community may rely on the common sense of its inhabitants to preserve order. However, as the community grows it may become helpful or indeed necessary to organise the unspoken rules into a formal system of law. The system should reflect the ‘common sense’ behaviour of the inhabitants; the fact that it has been written down should affect their daily lives very little.

Likewise, formalising a system into mathematical terms helps to keep order as the system becomes more complex. A mathematical system provides a framework for enquiry and argument when ‘common sense’ has been pushed to its limits; it does not otherwise interfere.

1.2 Category Theory: The mathematics of mathematics

It is not worth using an advanced classification system for a shelf of twenty books, but for a well-stocked library it is imperative.

I have asserted that mathematics is the rigorous study of conceptual systems, and that category theory is the mathematics of mathematics. So

category theory is the rigorous study of a conceptual system, where the system in question is mathematics itself.

Thus category theory may be seen as having two general roles:

1. To provide a language for making precise statements about *mathematical* concepts, and a system for making clear arguments about them.
2. To idealise *mathematical* concepts so that a diverse range of *mathematical* notions may be compared and studied simultaneously by focusing only on relevant features common to all of them.

I have already suggested that, to maintain rigour in a study, it is important to be precise about the context in which the issues are being considered. This is one of the principal considerations of category theory.

In everyday language, a category is some principle for grouping objects together and perhaps comparing them. It is difficult to compare objects without a context, or some criteria for making comparison. For example, a competition might be divided into categories, each having different criteria for judging entrants.

Likewise in mathematics, it is not enough to know which objects we are considering; we must also specify the context in which we are relating these objects. Is a bicycle better than an egg? In the category ‘transport’, a bicycle is clearly better, but certainly not in the category ‘food’.

A category, then, is a collection of objects together with some ways of relating them to each other.

1.3 Dimensions in Category Theory: Layers of complication

If an unknown animal is discovered, a theory of the unknown creature must be set up. It is clear that this theory is not itself an animal. However, if a mathematical concept is discovered, the theory of the new concept will itself be a new mathematical concept, itself requiring a theory ...

A category is a collection of objects together with some relationships between them. These relationships may also be regarded as objects and so might also have relationships between them. These relationships might also have relationships between them, which might have relationships between them ...

Each of these levels of ‘relationships’ is what is called a *dimension* in category theory. A basic category has only one level of ‘relationship’; it is a 1-dimensional category. If we allow relationships between relationships, we have a 2-dimensional category, or simply 2-category. Similarly we have 3-categories, 4-categories and so on; so we may have n -categories, where n is any whole number.

1.4 Higher Dimensional Category Theory: Minimal rules for maximal expression

A community without rules risks descending into chaos and disorder. But a strict regime suppresses expression and creativity.

‘Higher’ is a comparative term, so begs the question: higher than what? In this case, it means roughly ‘higher than what is easily defined’. What then are the difficulties in defining an n -category? The difficulty is in the rules.

Recall that a conceptual system is set up with two components: building blocks and rules. Higher dimensional category theory is no exception. An n -category has

- objects: called 0-cells
- relationships between objects: called 1-cells
- relationships between relationships between objects: called 2-cells
- relationships between relationships between relationships between objects: called 3-cells

... (all the above being building blocks)

- rules

The difficulty is that, just as relationships may have relationships, so rules may also satisfy rules. Rules for rules may also satisfy rules, and these themselves may satisfy more rules, and so on.

With insufficient rules, chaos would prevail, and the structure would be of little use. At the other extreme, it is easy to impose very strict rules, resulting in a highly disciplined and regimented system. So-called ‘strict n -categories’ are well understood.

However, such a strict regime does not allow the language to fulfill its expressive potential. So we need to find ‘minimal rules for maximal expression’.

The difficulty is that as the number of dimensions increases, the complexity of the necessary rules increases with fearsome rapidity. For 1 dimension, the rules may be written down on one line, and those for 2 dimensions may be expressed in diagrams occupying a page or so. For 4 dimensions the diagrams are already so large that they will not fit in any sensibly-sized book, and as such are unpublishable. The thought of writing down the rules for a 5-category would make most category theorists shudder, let alone for a 10-category or a 4-million-category.

Clearly, some other way of approaching the theory is required.

This is the great unsolved problem in higher-dimensional category theory: to make a general description of an n -category. Various different descriptions have been proposed by various category theorists with different aims and ideals. Different theories have been set up with different emphases, just as different languages have evolved to reflect different sensibilities. We need to be able to relate these theories to one another, like translating from one language into another.

Chapter 2

Completed research

The relationship between different approaches to higher-dimensional category theory.

There is an unmapped mountain. Various mountaineers claim to have reached the summit; each has returned with a map of his own route, and wondrous tales of the view from the top. To map the whole mountain we must at least see how the different routes relate to one another. Did all the mountaineers really reach the top? In fact, were they even climbing the same mountain?

The problem of defining an n -category has been approached in various different ways, but the relationship between the different approaches has not been fully understood. In many cases, one mathematician's attempt to understand an earlier definition has only resulted in his producing a new definition of his own. While it may be helpful to see many different facets of the same structure, such knowledge has only limited use if we do not know how the facets fit together.

Overview of my research to date

In 1997 John Baez and James Dolan proposed a definition of n -category ([BD]), called ‘opetopic’. Hermida, Makkai and Power attempted to understand it but, unable to do so, they attempted a definition of their own ([HMP]). Tom Leinster then attempted to understand the relationship between these approaches but, unable to do so, attempted yet another definition ([Lei]). Each definition had similar structure but used different ‘building blocks’.

In the first phase of my research ([Che1]) I succeed in providing a mathematical equivalence between the first two theories; in the second phase ([Che2]) I provide an equivalence between all three. (The notion of equivalence here has a highly rigorous mathematical definition.)

The third phase of my research involves proving that the proposed theory does not contradict the ‘evidence’ available. The notion of n -category is certainly understood when n is 0, 1, or 2; this is referred to as ‘classical’ theory. So for a theory of n -categories to be at all plausible, it must at least give the correct structures for these values of n . In ([Che3]) I prove that the opetopic theory is indeed equivalent to the classical theory for these low values of n .

2.1 The ‘opetopic’ definition of n -category

We have seen that a conceptual system is set up with two components: building blocks and rules. Then an n -category has

- Building blocks: 0-cells, 1-cells, 2-cells, 3-cells, ...
- Rules: rules for an n -category

The important thing to remember here is that n is *an arbitrary number*. We cannot simply write down the definition explicitly, because we do not know what that number is. So we need a general theory that will work *for all numbers*.

The definition proceeds as follows. First, a language for describing the cells is set up. Then, the rules are dealt with. So any comparison of the definitions must also proceed in these two stages. The language for describing the cells is the theory of *opetopes*.

2.2 Building blocks: opetopes

Like bricks, the building blocks in this theory, that is the cells, must be of certain ‘shapes’. An *opetope* (pronounced ‘op-e-tope’) is the proposed shape for a cell in this theory. Figure 2.1 shows some examples of typical opetopes in low dimensions.

The structures become rapidly complicated as the dimensions increase. The first problem, then, is to develop a convenient language for describing these objects, since drawing them is too impractical.

We may observe from the examples that a 2-dimensional opetope looks like a string of 1-dimensional opetopes ‘stuck together’ (ignoring arrowheads); similarly a 3-opetope looks like some 2-opetopes stuck together; a 4-opetope looks like some 3-opetopes stuck together, and so on. So we might simplify a 3-opetope, for example, by simply listing the 2-opetopes from which it was made, together with some sort of description of how they were stuck together.

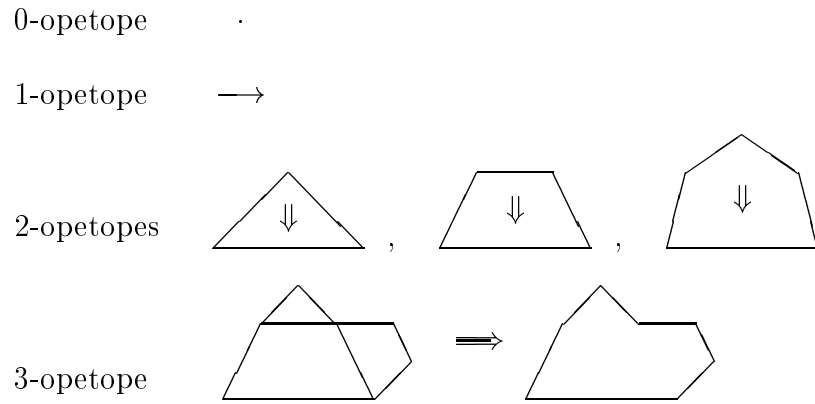
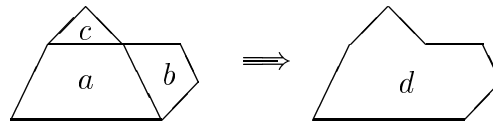


Figure 2.1: **Some typical opetopes of dimensions 0, 1, 2 and 3**

For example, labelling a 3-dimensional opetope as shown below:



we might then represent it as follows:

$$a, b, c \longrightarrow d.$$

This raises an immediate question: in what order should the components be listed? In the above case, we could equally have written

$$b, a, c \longrightarrow d$$

or

$$c, a, b \longrightarrow d,$$

to give just two of the alternatives.

Differences between the definitions

If we tidy up the papers on a desk, we force them into one pile even though they had natural positions strewn as they were. However, in a pile, they are easier to carry about.

It is somehow unnatural to try to force the components of an opetope into an orderly straight line when they have natural positions as they are; however, it would in many ways be much more practical to write them down in this way. But the problem is: *in what order should they be listed?*

The three different theories arise, essentially, from three different ways of tackling this issue. Baez and Dolan propose listing them in every order possible, giving one description *for each* ordering. Hermida, Makkai and Power propose picking one order at random. Leinster proposes *not* picking any order, but uses a much more complicated way of describing the picture without having to list the components at all.

The ‘differences’ are not differences

If we place an object in front of a mirror, we can see another image of the object, but we have not changed the object itself.

In fact, the Baez/Dolan approach as they present it is *not* equivalent to the other approaches. This is because they overlook an important matter in their scheme: if we are to have one description of a cell for *every* possible way of listing its components, we have many description of the same object, but we have not changed the object itself. Fatally, they disregard, or ‘throw away’ the fact that these are images of the same object, and so a huge number

of extra objects seem to have appeared from nowhere. But, like reflections in a mirror, they are not really there.

My first task was therefore to modify their theory to preserve the important facts they had abandoned. Once this was done, I was able to prove that the three different theories were in fact mathematically equivalent.

2.3 Rules

Every lock has a key that fits it best.

The next step was then to consider the ‘rules’ for the system. Since Baez and Dolan had set up their rules with a crucial oversight in their building blocks, the first problem was to modify the rules to take into account the *modified* building blocks. Only then could comparisons proceed.

The idea is that an n -category has ‘spaces’ where cells can be. The rules are stated in the form ‘every space for a cell has a cell that fits best’. Compare this with ‘every parking space has a car that fits in it best’ or ‘every lock has a key that fits it best’. The first statement is dubious, but the second is plausible.

Hermida/Makkai/Power and Leinster did not complete their definitions to include the rules, so the next step was to compare the modified Baez/Dolan definition with the so-called ‘classical theory’, that is, the well-understood low-dimensional cases. Baez and Dolan themselves checked their definition for $n = 1$ but did not attempt a comparison for $n = 2$. Perhaps this is not surprising since, without my modification to their theory, such a comparison was impossible.

Differences between the definitions

Prince Charming knows that someone's foot fits the glass slipper, but this is not enough for him: he must find her.

The most notable difference between the opetopic and classical theories is that in the classical theory one particular ‘best cell’ is actually specified for each space; in the opetopic theory the best cells are simply known to exist. The problem is that there may be more than one possible ‘best cell’, just as there may be several copies of a key. To compare the theories, which of the possible ‘best cells’ are we to choose? What would Prince Charming have done if two people’s feet had fitted the glass slipper?

The ‘differences’ are not differences

If we are given a bunch of identical keys to a door, we can choose any of them to unlock the door. It does not matter which one we choose, and the act of choosing does not change the bunch of keys in our hands.

In my most recent work, I prove that the opetopic and classical approaches are equivalent for $n = 2$, despite the issues discussed above. This equivalence is somewhat surprising, as questions of choice usually complicate matters a great deal.

However, I show that, even if there is a chosen best cell for a space, it makes no difference, as long as we can still work out what the other candidates for selection *could have been*.

Chapter 3

The future: up and along

As I progress towards the summit of the mountain I must at each stage decide whether to proceed straight up the face or to edge my way further around.

There are two general directions in which I propose to continue my research.

3.1 ‘Up’

The opetopic theory of n -categories is still only partly complete, although the definition of an n -category itself is in place. At low dimensions, there is the possibility of one more level of comparison with the classical theory: the ‘classical’ theory of 3-categories is well-established, albeit not as well-known as that of 2-categories. This is the next issue begging to be addressed.

Remaining, for the time being, in low dimensions, I might then examine other aspects of the classical theory with a view to ‘translating’ them into the opetopic world. In particular, the so-called ‘coherence theorems’, which

formalise the crucial assertion ‘this theory is sensible’. Coherence for 2-dimensions is easy to state and relatively straightforward to prove. As is to be expected, coherence for 3-dimensions is much more complex; for higher dimensions we should expect very complex notions indeed. It would therefore be of great value to be able to understand the coherence issue in a general n -dimensional setting, without having to consider each dimension separately. The opetopic theory is potentially such a setting.

As for the higher dimensions, there remains the important idea that the collection of n -categories should itself form an $(n+1)$ -category. Certainly, the collection of 1-categories forms a 2-category, and the collection of 2-categories forms a 3-category. The reasons for this are sufficiently clear that the need for a generalisation to n is indisputable. It is therefore of utmost importance that this matter be resolved in order for any theory of n -categories to be at all satisfactory. In the opetopic theory, this still remains to be done.

3.2 ‘Along’

There are now approximately ten theories of n -categories proposed in the literature, although some are spin-offs of others. It is crucial that these theories be compared, and yet the process of comparison is not, to my knowledge, being systematically undertaken. It is possible that many mathematicians in the field are too well versed in one particular approach to be able fully and objectively to consider any other. Whatever the case, I propose to examine each of these definitions and, one by one, compare them with the theories already unified. My eventual aim is to present a complete theory of n -categories which is more than just the sum total of all proposals to date; as

in the low-dimensional cases, it should actually cover all the possible theories that could ever be proposed.

It is not enough to climb the mountain; we must map the mountain, so that others may ascend and admire the breathtaking view themselves.

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Category theory, higher-dimensional category theory, model categories, triangulated categories, operads, applications. Differential geometry Lie groupoids and Lie algebroids, foliation theory, étale groupoids, orbifolds; Number theory elliptic curves, modular forms, Eisenstein cohomology. 3 Higher dimensional algebra. The aim here is to explain some mathematical ideas with which the authors have been preoccupied since the 1960s and 1970s respectively. There was a lot of experimentation to produce the mathematics which would encompass some apparently simple intuitions. This experimentation can be viewed as "extraction of concepts", and it also exemplifies mathematical concepts that might provide a model of inter-relationships that is much freer than the usual ones. Indeed, higher dimensional algebra is being used to model distributed systems. and the language for expressing this is again that of category theory. It is useful to express this intuition as "composition is an algebraic inverse to subdivision". The labelled subdivided line gives the composite word, $abcd$. Higher-dimensional categories are like a vast mountain that many people are trying to conquer. Some intrepid explorers have made the ascent, each taking a different route and each encountering different hazards. Each has made a map of his route, but do we know how all these maps fit together? Do we know that they fit together at all? In fact, are we even climbing the same mountain? This has involved shifting our mind-set around as we move between different theories, a sort of mental gymnastics that we feel is an essential warm-up for the exercise of understanding. This work grew out of a very real situation: the second named author asked the first for an introduction to the various definitions of n -category, in advance of the IMA n -categories workshop in June 2004.