MOMENT MAPS IN VARIOUS GEOMETRIES

L. JEFFREY (TORONTO), T. HOLM (BERKELEY), Y. KARSHON (TORONTO), E. LERMAN (CANBERRA / URBANA-CHAMPAIGN), AND E. MEINRENKEN (TORONTO)

Abstract. This is the final report on the 5-day workshop on *Moment maps in various geometries*, held at the BIRS from May 21 to May 26.

1. Background

Symplectic geometry was invented by Hamilton in the early nineteenth century, as a mathematical framework for both classical mechanics and geometrical optics. Physical states in both settings are described by points in an appropriate phase space (the space of coordinates and momenta). Hamilton's equations associate to any energy function ("Hamiltonian") on the phase space a dynamical system. Hamilton realized that his equations are invariant under a very large group of symmetries, called canonical transformations or, in modern terminology, symplectomorphisms. A symplectic manifold is a space which is locally modeled by the phase spaces considered by Hamilton. In mathematical terms, a symplectic manifold is a manifold M with a closed, non-degenerate 2-form ω. A smooth function $H \in C^\infty(M)$ defines a vector field X_H on M by Hamilton's equations,

$$\mathrm{d}H = -\omega(X_H, \cdot).$$

New techniques have transformed symplectic geometry into a deep and powerful subject of pure mathematics. One concept of symplectic geometry that has proved particularly useful in many areas of mathematics is the notion of a *moment map.* To recall the original setting for this notion, let M be a symplectic manifold, and G a Lie group acting on M by symplectomorphisms. A moment map for this action is an equivariant map $\Phi\colon M \to \mathfrak{g}^*$ with values in the dual of the Lie algebra, with the property that the infinitesimal generators of the action, corresponding to Lie algebra elements $\xi \in \mathfrak{g}$, are the Hamiltonian vector fields $X_{\langle \Phi, \xi \rangle}$. The linear momentum and angular momentum from classical mechanics may be viewed as moment maps, corresponding to translational and rotational symmetries, respectively.

In the past thirty years, tremendous progress has been made in the study of moment maps and related areas: symplectic quotients, geometric quantization, localization phenomena, and toric varieties. This has had applications to the study of moduli spaces, representation theory, special metrics, and symplectic topology.

In recent years, moment maps have been generalized in many different directions and have led to advances in geometries related to symplectic geometry. These include Poisson geometry, Kähler geometry, hyper-Kähler geometry, contact geometry, and Sasakian geometry. While some headway has been made in understanding moment maps in these fields, there remain many open questions. One of the goals of this workshop was to explore phenomena that are well understood in symplectic geometry but are not as well understood in these new settings, and to seek potential applications of this new direction of research. For this purpose we brought together experts from these fields, thus generating a fruitful exchange of ideas, which also enabled us to formulate and discuss interesting open problems.

2. Objectives of the workshop

Let us review some of the achievements in and applications of equivariant symplectic geometry in the past few years. We will then indicate some of the open questions that were our motivation for holding the workshop.

We first recall some terminology. Let a Lie group G act on a symplectic manifold (M, ω). As we already recalled, a *moment map* is an equivariant map $\Phi\colon M \to \mathfrak{g}^*$ to the dual of the Lie algebra such that the G action is generated by the Hamiltonian vector fields of the components of Φ. The *symplectic quotient* is $\Phi^{-1}(0)/G$. *Localization* formulas express global invariants of M in terms of local data at the fixed point set of an abelian subgroup of G. When G is a torus of half the dimension of M and M is compact, (M, ω, Φ) is a *toric manifold*.

A *contact structure* is an odd dimensional counterpart of a symplectic structure. Similarly, a *Sasakian structure* is an odd dimensional counterpart of a Kähler structure, and a *3-Sasakian structure* is an odd dimensional counterpart of a hyper-Kähler structure. The goal of the workshop was to obtain a better understanding of moment maps and their applications in these other geometries.

The development of equivariant symplectic geometry over the last two decades was greatly motivated by attempts to understand the topology of moduli spaces of stable bundles over Riemann surfaces. The symplectic and Morse theoretic approach to the problem was pioneered by Atiyah and Bott in 1983, when they produced a set of generators for the cohomology ring of the moduli space $M(n, d)$ of semi-stable rank n, degree d holomorphic vector bundles over a Riemann surface, for n and d co-prime.

Given a Hamiltonian group action of a Lie group G on a compact symplectic manifold M, with moment map $\Phi\colon M \to \mathfrak{g}^*$ such that 0 is a regular value for Φ, there is a natural map from the equivariant cohomology $H^*_G(M)$ to the cohomology of the reduced space, $H^*(\Phi^{-1}(0)/G)$, obtained as the restriction $H^*_G(M) \to H^*_G(\Phi^{-1}(0))$ followed by the isomorphism $H^*_G(\Phi^{-1}(0)) \to H^*(\Phi^{-1}(0)/G)$. Kirwan refined the Morse-theoretic methods of Atiyah and Bott to prove that this map, $\kappa\colon H^*_G(M) \to H^*(\Phi^{-1}(0)/G)$, is surjective. This enables one to compute Betti numbers of symplectic quotients $\Phi^{-1}(0)/G$. The non-abelian localization theorem of Jeffrey and Kirwan gives an explicit formula for the kernel of κ. Jeffrey and Kirwan used their version of the non-abelian localization formula, and a description of $M(n, d)$ as a finite-dimensional quotient of a so-called "extended moduli space", to obtain a mathematically rigorous proof of Witten's formulas for the intersection pairings in the cohomology of $M(n, d)$.

In 1998, Alekseev, Malkin and Meinrenken introduced quasi-Hamiltonian spaces and Lie group valued moment maps. They expressed the moduli space of flat G-connections as a quasi-Hamiltonian quotient of a product $G^2 \times \cdots \times G^2$, and were thus able to recover Witten's formulas for intersection numbers in the cohomology of moduli spaces. In the moduli space case, quasi-Hamiltonian spaces enable one to avoid the use of extended moduli spaces; more generally, quasi-Hamiltonian spaces enlarge the collection of situations to which similar techniques can be applied.

In 2002, Bott, Tolman and Weitsman proved surjectivity of Kirwan's map $\kappa : H^*_{LG}(M) \to H^*(\Phi^{-1}(0)/G)$ in the case where LG is the loop group of a compact Lie group G, M is a Banach manifold and Φ a proper moment map. As a consequence one obtains that, while Kirwan's map is not surjective for quasi-Hamiltonian spaces, its image together with *finitely many* cohomology classes generates the cohomology ring of the quotient. This work is related to Tolman and Weitsman's earlier work (1998) determining the kernel of the Kirwan map κ and thereby the structure of the cohomology ring of the symplectic quotient $H^*(\Phi^{-1}(0)/G)$ when G is a finite-dimensional Lie group.

In 2003, Xu introduced quasi-symplectic groupoids. This approach enables him to unify into a single framework various moment map theories, including ordinary symplectic moment maps and group valued moment maps.

Moment maps and symplectic quotients can be defined in other categories, such as contact or hyper-Kähler. However, the topology of quotients in these categories remains elusive. As noted in a recent book by Ginzburg, Guillemin, and Karshon, phenomena such as Kirwan surjectivity and localization are often due to the underlying moment map and group action more than to the geometry. However, we do not yet understand these phenomena for contact or hyper-Kähler manifolds. For example, Kirwan surjectivity fails for contact structures, and it is not yet clear why or how. Surjectivity is conjectured for hyper-Kähler quotients, and known to be true for many classes of examples, but a general theorem has not been proved. An interesting example of a hyper-Kähler quotient is the moduli space of rank 2 parabolic Higgs bundles. Hausel and Thaddeus have produced generators and relations for the cohomology ring of this space. This work is analogous to the work of Jeffrey and Kirwan on the moduli space $M(n, d)$. Another usage of hyper-Kähler quotients is that they provide examples of Einstein manifolds.

In 1988 Delzant classified symplectic toric manifolds. These turn out to be symplectic quotients of $\mathbb{C}^N$. In particular, they inherit a complex structure from $\mathbb{C}^N$, making them into smooth Kähler toric varieties. The images of their moment maps are simple rational polytopes satisfying certain integrality conditions. The polytope determines the toric manifold together with its symplectic form and torus actions. This theorem of Delzant, while simple in retrospect, inspired a lot of interesting mathematics. For example, the removal of the integrality condition on simple rational polytopes leads to orbifold singularities. Symplectic toric orbifolds were classified in 1997 by Lerman and Tolman in terms of simple rational polytopes with positive integers attached to facets. Delzant's work inspired Banyaga and Molino to initiate the study of *contact* toric manifolds. The classification of contact toric manifolds has been recently completed by Lerman. Lerman used this classification to prove conjectures of Toth and Zelditch on toric integrable geodesic flows. Most, but not all, of the contact toric manifolds turn out to be Sasakian. These contact toric manifolds are classified by rational polyhedral cones.

Yet another direction inspired by Delzant's work is that of hyper-Kähler toric manifolds. These manifolds were first studied by Bielawski and Dancer, who defined them to be hyper-Kähler quotients of a flat quaternionic vector space. They obtained a formula for the Betti numbers of these manifolds in terms of the corresponding arrangements of hyperplanes. Bielawski also showed that these are all complete hyper-Kähler manifolds with torus symmetries of maximal dimension. At the same time, Bielawski obtained a classification of toric 3-Sasakian manifolds. In 2000, Konno computed the full cohomology ring of a hyper-Kähler toric manifold in terms of the hyperplane arrangement. In a later paper Konno computed the total Chern classes of these manifolds.

An important use of toric varieties, in both complex and symplectic geometry, is to provides a large "hands-on" family of examples. In particular, they have been used in searches for examples of special Kähler metrics.

A formula for the Kähler metric on a toric manifold, in terms of natural linear functions on the polytope, was obtained by Guillemin in 1994. Guillemin's work, in turn, inspired Abreu, who studied other metrics on symplectic toric manifolds. For example, Abreu obtained an explicit description of Bochner-Kähler metrics studied by Bryant. He also obtained a combinatorial formula for their scalar curvature and used it to explicitly construct Kähler metrics that are extremal in the sense of Calabi. One question that remains open is to obtain explicit formulas for Kähler-Einstein metrics on $\mathbb{CP}^2$ blown up at three generic points; such metrics are only known to exist.

Recently a great deal of progress has been made by Boyer, Galicki and their collaborators in proving the existence of Sasakian-Einstein metrics on a large class of contact manifolds. These metrics, however, are not known explicitly. One expects that an analogue of Guillemin's formula for Kähler metrics on symplectic toric manifolds to hold for the Sasakian toric manifolds. These metrics are unlikely to be Einstein (this follows from very recent work of Guillemin and Burns). However, it might be possible to construct the Sasakian-Einstein metrics explicitly in terms of polyhedral cones.

There have been a variety of other applications of moment maps to the study of special metrics. Futaki and Tian used localization to compute an invariant which provides an obstruction to the existence of constant scalar curvature metrics in a fixed Kähler class. For a toric variety, Mabuchi expressed this invariant in terms of the corresponding polytope. Claude Lebrun and Michael Singer used moment maps to explore scalar-flat Kähler metrics on ruled surfaces. "Extremal" metrics and "central" metrics are ones for which certain elementary symmetric functions of the Ricci curvature are moment maps for Killing fields. An outstanding conjecture is whether the existence of constant scalar curvature metrics, or Kähler-Einstein metrics, is equivalent to certain notions of "stability". Results in this direction have been obtained by Tian (1997) and Tian-Chen (as announced very recently). Another part of this conjecture was recently proved by Donaldson for the special case of toric manifolds in complex dimension 2. In a different direction, one can exhibit the scalar curvature as a moment map in an infinite dimensional setting. This description is due to Mabuchi and was used by Donaldson. It is analogous to Atiyah and Bott's influential work on the Yang Mills functional.

One of our motivating goals was to determine which invariants developed in symplectic geometry for understanding symplectic quotients (for example their cohomology ring) carry over to the settings of hyper-Kähler, contact, Sasakian, and 3-Sasakian geometries. In particular, we proposed to explore the question of surjectivity in contact and hyper-Kähler geometry. Additionally, we aimed to study natural metrics on such quotients and to use this to seek explicit descriptions for special metrics on Kähler and Sasakian manifolds.

At the workshop, besides an under-representation of the odd dimensional structures (contact, Sasakian, 3-Sasakian), the lectures and discussions addressed many aspects of moment maps in a wide variety of contexts: Kähler geometry and special metrics, applications to symplectic topology, approaches through Lie groupoids, algebraic geometric, several aspects of hyper-Kähler geometry, and more.

3. Activities of the workshop

The formal activities during the workshop included research talks, survey lectures on special topics, and two problem sessions, aimed as forums for discussion. We believe that this format has been highly successful and very stimulating. Below, we will summarize some of the new developments and open questions presented at the workshop.

Moment maps and symplectomorphism groups

Let (M, ω) be a symplectic manifold, and $\mathrm{Diff}_\omega(M)$ its group of symplectomorphisms. The group $\mathrm{Diff}_\omega(M)$ contains an important subgroup $\mathrm{Diff}_{\mathrm{Ham}}(M)$ of *Hamiltonian diffeomorphisms*, i.e., the subgroup generated by time-one flows of Hamiltonian vector fields. The topology of the groups $\mathrm{Diff}_{\mathrm{Ham}}(M)$ and $\mathrm{Diff}_\omega(M)$ has been the subject of intense research over the past few years.

Miguel Abreu (Instituto Superior Tecnico, Lisbon) (joint work with Granja and Kitchloo) reported on recent progress on the topology of $\mathrm{Diff}_\omega(M)$. The basic new input goes back to Donaldson, and uses the moment map geometry for the action of a symplectomorphism group on the space of compatible almost complex structures. In conjunction with his earlier work [1] with McDuff, employing Gromov's technique of pseudo-holomorphic curves, this approach turns out to be particularly successful for a class of 4-dimensional symplectic manifolds, including rational ruled surfaces.

Susan Tolman (University of Illinois at Urbana-Champaign) (joint work with McDuff) described exciting new results on the fundamental group of symplectomorphism groups of 4-dimensional symplectic toric varieties M, i.e., spaces carrying an effective Hamiltonian action of a torus of dimension $\frac{1}{2}\dim M = 2$. A well-known theorem of Delzant (see e.g. [11]) states that such spaces are completely determined (up to equivariant symplectomorphism) by the convex polytope in $\mathbb{R}^2$ given as their moment map image. Moreover, one can specify exactly which polytopes arise as moment polytopes of Delzant spaces. In their work, McDuff-Tolman discovered a relationship between the topology of the symplectomorphism group of such spaces with the shape of the moment polytope. This then leads to the following problem: Which Delzant polytopes admit a linear function so that the center of mass of the polytope depends linearly on the facet position? The solution to this problem allows them to prove that, for all but a few exceptional cases, the inclusion of the (compact) group of Kähler automorphism into the group of symplectomorphism induces an isomorphism of fundamental groups.

Victor Guillemin (M.I.T.) (joint work with Sternberg) described a very different aspect of symplectomorphism groups. He explained that for certain maps from finite dimensional manifolds into the group of symplectomorphisms, there is an intriguing notion of a moment map even if there is no Hamiltonian group action! In his beautiful talk, he motivated how this type of generalized moment map fits with Weinsteins *symplectic category* [27]. This is the "category" with objects Obj symplectic manifolds M, and morphisms $\mathrm{Mor}(M_1, M_2)$ the canonical relations, meaning, Lagrangian submanifolds of $M_1^- \times M_2$. (Here "category" is put in quotes, since composition is not always defined.) Concrete applications of this theory arise in micro-local analysis, in the study of families of Fourier integral operators.

Moment maps and Poisson geometry

Poisson manifolds are manifolds M equipped with a Poisson bracket $\{\cdot,\cdot\}$ on the algebra of smooth functions on M. Symplectic manifolds are special cases of Poisson manifolds, where the bracket is given as

$$\{f, g\} = X_f(g).$$

A Poisson structure determines a singular foliation (in the sense of Sussmann) whose leaves are symplectic manifolds.

Rui Fernandes (Instituto Superior Tecnico, Lisbon) (joint work with Crainic). The Poisson bracket descends to a canonical Lie bracket on the space of 1-forms on any Poisson manifold. In this way, the cotangent bundle T^*M acquires the structure of a *Lie algebroid*. A global object 'integrating' this Lie algebroid is a *symplectic groupoid*, i.e., a groupoid

$$S \rightrightarrows M,$$

where S carries a symplectic structure such that both groupoid maps are Poisson maps, and such that the symplectic form is compatible with the groupoid multiplication. Not all Poisson manifolds admit such a *symplectic realization*. The precise obstructions were found a few years ago by Fernandes-Crainic [10]. In his BIRS lecture, Fernandes explained how this theory extends to the presence of Poisson group actions. He showed that if M admits a symplectic realization S, then the induced action on S is Hamiltonian with a canonical moment map. (This moment map satisfies a cocycle condition, and is a coboundary if and only if the action on M admits a moment map.) Finally, Fernandez explained in which sense 'symplectic realization' commutes with 'reduction'.

Anton Alekseev (University of Geneva) (joint work with Meinrenken [3]). A *Poisson Lie group* is a Lie group K with a Poisson structure for which the product map is Poisson. This condition defines a Lie bracket on the dual of the Lie algebra $\mathfrak{k}^*$, which integrates to the so-called *dual Poisson Lie group* K^*. If K carries the zero Poisson structure, then the dual Poisson Lie group is $\mathfrak{k}^*$ with the Kirillov Poisson structure. A construction of Lu-Weinstein [23] shows that any compact Lie group K admits a canonical Poisson Lie group structure. Later, Ginzburg-Weinstein [14] proved that, in this case, the dual Poisson Lie group K^* is Poisson diffeomorphic to $\mathfrak{k}^*$. However, no explicit form of such a diffeomorphism was known. Alekseev explained that for the group $K = \mathrm{U}(n)$, there is a distinguished and very concrete Ginzburg-Weinstein diffeomorphism

$$\mathfrak{u}(n)^* \to \mathrm{U}(n)^*.$$

The proof of this result (which verifies a conjecture of Flaschka-Ratiu [13]) is based on a study of Gelfand-Zeitlin systems on $\mathfrak{u}(n)^*$ and $U(n)^*$, respectively. As a corollary, one obtains the following interesting result: There is a canonical diffeomorphism

$$\gamma\colon\ \mathrm{Herm}(n) \to \mathrm{Herm}^+(n)$$

from hermitian matrices to positive definite Hermitian matrices, with the property that the eigenvalues of the kth principal submatrix of $\gamma(A)$ are the exponentials of those of the kth principal submatrix of A.

Groupoids and generalized moment maps

Markus Pflaum (Goethe Universität, Germany) Differentiable groupoids can be interpreted as an interpolation between the notion of a manifold and the notion of a Lie group. In this survey talk, Markus Pflaum gave a general introduction to the theory of Lie groupoids (cf. [12]), and explained two major applications of this theory in symplectic geometry. The first application deals with the integrability of Poisson manifolds by symplectic groupoids (cf. Fernandes' lecture). The second application is Moerdijk's approach [24] to orbifolds via proper étale Lie groupoids, which is an important ingrediant in the work by Pflaum–Posthuma–Tang on the deformation quantization and index theory for orbifolds.

Henrique Bursztyn (University of Toronto) presented a survey lecture on generalized moment maps (cf. [9]). He explained how, quite generally, any Poisson map between Poisson manifolds defines an infinitesimal 'Lie algebroid' action, and hence may be viewed as a moment map. This includes ordinary $\mathfrak{k}^*$-valued moment maps, but also Lu's [22] non-linear moment maps taking values in a dual Poisson Lie group K^*. To include more exotic types of moment maps, one has to go beyond Poisson structures to so-called *twisted Dirac structures*. In particular, any compact Lie group carries a natural twisted Dirac structure, and the associated moment map theory defines the q-hamiltonian spaces of Alekseev-Malkin-Meinrenken [2]. Among the advantages of this approach is that the somewhat mysterious 'minimal degeneracy conditions' become very natural. Furthermore, the techniques work well also for non-compact Lie groups, as well as for complex Lie groups.

Topology of symplectic quotients

Let (M, ω) be a symplectic manifold, equipped with a Hamiltonian action of a Lie group K, with moment map Φ. A standard result of Marsden-Weinstein asserts that under suitable assumptions, the *symplectic quotient*

$$M/\!\!/G = \Phi^{-1}(0)/G$$

inherits a symplectic structure from the 2-form on M. It is a fundamental problem in symplectic geometry to understand the geometry and topology of $M/\!\!/G$ in terms of the equivariant geometry of the original space M.

Greg Landweber (University of Oregon) (joint work with Harada [18]). In this survey lecture, Landweber gave a general overview of equivariant K-theory (the generalized cohomology theory given as the Grothendieck ring of equivariant vector bundles) in the context of Hamiltonian group actions. He explained the K-theory analog of the Atiyah-Bott Lemma, which says that the K-theory analogue of the equivariant Euler class is not a zero divisor. As a result, one obtains a K-theoretic analogue of the Kirwan surjectivity theorem. As Landweber remarks, the torsion in K-theory is better behaved than that in cohomology with integer coefficients. Essentially, K-theory eliminates just enough torsion for Atiyah and Bott's arguments to work.

Liviu Mare (University of Regina) (joint with Harada, Holm and Jeffrey [17]). Classical results of Atiyah [6], Guillemin-Sternberg [15] and Kirwan [19] say that for any compact torus T, and any Hamiltonian T-space with proper moment map, the image of the moment map is a convex polyhedron, and the fibers of the moment map are connected. Atiyah-Pressley [8] proved a similar convexity result for the maximal torus $\tilde{T}$ in the standard extension of the based loop group ΩG for a compact, simply connected Lie group. The main result presented in this lecture says that also in this case, the fibers of the moment map are connected.

Nick Proudfoot (UT Austin) ([25]) Suppose G is a reductive algebraic group, acting on a variety Q. Then the cotangent bundle T^*Q has an algebraic symplectic form, and the lifted G-action is Hamiltonian with an algebraic moment map. In his talk, Proudfoot discussed the relation between the symplectic quotient of T^*Q, with various GIT (geometric invariant theory) quotients of Q.

Kähler geometry and special metrics

A Kähler manifold is a manifold with compatible complex and symplectic reduction. The presence of a complex structure leads to stronger versions of some of the results of moment map geometry.

Reyer Sjamaar (Cornell University) (joint work with V. Guillemin [16]). For Hamiltonian torus actions on *Kähler manifolds*, Atiyah [6] had proved an important refinement of the convexity theorem: Not only is the image of the moment map a convex polytope, but in fact the moment map image of any orbit closure is convex. (Note that orbit closures need not be smooth submanifolds.) Brion generalized the result to actions of a complex reductive group. The results presented in this lecture generalize this result even further, to actions of a maximal solvable subgroup. Two interesting examples of Borel-invariant subvarieties of a Hamiltonian Kähler G-manifold are: (1) Generalized

Schubert varieties (introduced by Białnicky-Birula, and (2) the co-called facial varieties. That is, for each face of the moment polytope there is a certain variety whose moment map image is the given face. (In general, there is no G-invariant subvariety with this property.)

Vestislav Apostolov (UQAM) (joint work with Calderbank, Gauduchon, and Tonnesen-Friedman [5]). In recent work, Apostolov and his coauthors introduced the notion of *Hamiltonian 2-forms on Kähler manifolds.* These are closed differential forms of bidegree $(1,1)$, defined as solutions of a certain linear differential equation on the Kähler manifold. Hamiltonian 2-forms arise, for example, in the theory of Bochner-flat or conformally Einstein Kähler manifolds. Apostolov's lecture was concerned with the local and global classification of Hamiltonian 2-forms. As applications, he obtained new examples of so-called *orthotoric Kähler-Einstein manifolds.*

Hyper-Kähler geometry

Hiroshi Konno (University of Tokyo) gave a survey lecture on the geometry and topolgy of hyper-Kähler quotients. Examples for such quotients include: toric hyper-Kähler manifolds, hyper-Kähler polygon spaces, the moduli space of torsion free sheaves on $\mathbb{C}^2$, and Nakajima quiver varieties.

Tamas Hausel (UT Austin) explained techniques for the computation of cohomology groups of hyper-Kähler manifolds, such as moduli space of instantons, quiver varieties, representation varieties, and moduli of Higgs bundles. The techniques are: (i) global analysis to determine the space of L^2-harmonic forms (this approach is motivated by Sen's conjecture); (ii) circle-equivariant cohomology techniques (motivated by ideas of Nekrasov-Shatashvili-Moore) and (iii) calculation of zeta functions by arithmetic harmonic analysis (motivated by mirror symmetry).

Graeme Wilkin (Brown University) (joint work with Daskalopoulos and Wentworth). In their 1982 paper, Atiyah and Bott [7] used Morse theory of the Yang-Mills functional to study the topology of the moduli space of semistable vector bundles over a Riemann surface. Wilkin described a similar technique for the moduli space of rank 2 semi-stable Higgs bundles. A complication in this example is that the moduli spaces are singular, and hence the method has to refined to take the singularities into account. A main result of this approach is a proof of Kirwan hyper-Kähler surjectivity for some rank-2 Higgs bundles.

Moment maps and path integrals

Jonathan Weitsman (Santa Cruz). Quantum field theory is a source for many exciting predictions in mathematics, mostly based however on non-rigorous 'functional integral teachniques'. A prototype is Witten's formulas [28] for intersection pairings, based on path integral calculations for the Yang-Mills functional (norm square of the moment map). In his talk, Weitsman indicated that in some case, these path integral arguments can in fact be made rigorous. The main techique is a new construction of measures on Banach manifolds associated to supersymmetric quantum field theories. As examples, he discussed the Wess-Zumino-Novikov-Witten model for maps of Riemann surfaces into compact Lie groups, as well as 3-dimensional gauge theory.

4. Open problems

In addition to the traditional lectures, we ran two problem sessions during our week at Banff. These sessions were meant to foster discussion and to identify open problems relevant to the workshop. Each session had a moderator who solicited the open problems from the audience and transcribed them onto the board. We used a format very similar to the problem sessions run at the workshops at the American Institute of Mathematics [4]. We present here the record of the problems discussed at these sessions.

4.1. Compactification of cotangent bundles.

Problem 4.1 (N. Kitchloo)**.** *Let X be a compact manifold. Does the symplectic manifold (T^*X, ω) have a "natural" compactification $(Y, \tilde{\omega})$ so that $\tilde{\omega}|_{T^*X} = \omega$?*

Several commented that this question is a bit misleading, since T^*X has infinite volume. We may modify it to ask about the disc bundle in T^*X.

Nick Proudfoot pointed out that this is equivalent to asking whether or not X is a Lagrangian submanifold of some compact symplectic manifold.

Eugene Lerman noted that this is true for $X = S^3$, and is true more generally if X is a Riemannian manifold with a periodic geodesic flow: then we may "cut" T^*X with respect to the energy functional. For example, we may do this when $X = S^3$ or when X is a Zoll surface. Of course, if not all periods are the same, one may end up with an orbifold.

If X is a complex manifold, there is a natural S^1 action on the fibres; however, this action is not symplectic.

Allen Knutson commented about the case when X is a real algebraic variety. Then X is the real locus of $X(\mathbb{C})$, a complex algebraic variety. Let Y be a desingularization of the closure of $X(\mathbb{C})$ in projective space. Note that the singularities are all far from X. Then X still sits inside as a Lagrangian submanifold.

Eugene Lerman pointed out that we may take Y to be Thom space of T^*X or the one-point compactification. If we view this as a symplectic stratified space, X is a Lagrangian submanifold. This may not be "natural".

Markus Pflaum mentioned that a similar question was addressed in [21].

4.2. Circle actions and the Hard Lefschetz Property.

Let (M, ω) be a $2n$-dimensional compact symplectic manifold. Consider the map

$$\begin{aligned} L : H^i(M) &\rightarrow H^{i+2}(M) \\ c &\mapsto [\omega] \cup c. \end{aligned}$$

We say that M satisfies the *Hard Lefschetz property* if

$$L^k \colon H^{n-k}(M) \rightarrow H^{n+k}(M)$$

is an isomorphism for each $0 \leq k \leq n$.

Participants note: All compact Kähler manifolds satisfy Hard Lefschetz. Specifically, if M is a projective variety, then ω is the restriction of the Fubini-Study form on projective space, so the Kähler class is the Poincaré dual of a hyperplane section. So L is the intersection with this hyperplane section, and Hard Lefschetz holds.

Problem 4.2 (Y. Karshon)**.** *Suppose that (M, ω) admits a Hamiltonian S^1 action with isolated fixed points. Does (M, ω) satisfy the Hard Lefschetz property?*

This problem has been around for at least 13-14 years; Yael isn't sure of its origin.

Reyer Sjamaar comments that his student Yi Lin has worked on a related question. Symplectic quotients often inherit nice properties from the original manifold: if the original manifold is Kähler, so is its symlectic quotient. Yi Lin has shown that symplectic quotients **do not** inherit the Hard Lefschetz property.

Nick Proudfoot asked why having an action should say anything about Hard Lefschetz. Yael Karshon replied that having a Hamiltonian action with isolated fixed points is a very strong assumption.

Reyer Sjamaar pointed out that, by a result of Susan Tolman and Jonathan Weitsman, if the S^1 action is in addition semi-free, then $H^*(M)$ is isomorphic as a ring to $H^*((P^1)^k)$. Under the isomorphism, $[\omega]$ maps to the class that is the product of Fubini-Study 2-forms and takes the first Chern class to the first Chern class. Thus the Hard Lefschetz property holds for these examples.

Nitu Kitchloo asked if it makes any difference if ω is integral. Then we may classify ω by a map to $\mathbb{C}P^\infty$. This gives a principle S^1 bundle P over M, and Hard Lefschetz is equivalent to $H^*(P)$ being a "very small" cohomology ring. This follows from the Leray-Serre spectral sequence for the cohomology of the total space.

Sue Tolman points out that an easier version of this problem is as follows.

Problem 4.3 (S. Tolman)**.** *Are the Betti numbers of M unimodal? That is, do they satisfy*

$$\beta_1 \le \beta_3 \le \cdots \le \beta_{half}$$

and

$$\beta_2 \le \beta_4 \le \cdots \le \beta_{half}?$$

4.3. **$\mathbb{Z}_2$-graded ("super") symplectic manifolds and reduction.** Let M be a $\mathbb{Z}_2$-graded symplectic manifold (for a reference, see [20]). That is, M is locally a manifold, and over each open set U, we have a trivial bundle $E = V \times U$. The "functions" on U are $C^\infty(U) \otimes \Lambda^*(V)$. The "odd" variables live in "flat" directions corresponding to V ("ectoplasm has no topology!"). This is one way to define a super manifold. Extend this to global structure by patching. A **symplectic form** in this setting is anti-symmetric on the even (standard) directions and symmetric on the odd (V) directions.

Consider the case where the space of odd variable seems NOT flat. Take $M = pt$. Then we have only an odd vector space V and the "symplectic form" is a Euclidean metric (inner product). For $G \subseteq SO(V)$ acting, we can define a moment map. It seems that the "symplectic quotient" will not necessarily be a $\mathbb{Z}_2$-graded symplectic manifold in the above sense.

Now consider the "quantization," which is the space of functions on the manifold. This is the spinor representation $\mathbb{S}(V)$ of the Clifford algebra of V. Now restrict to the G-invariant part to "reduce" the "quantization."

Problem 4.4 (S. Wu)**.** *What is the classical analogue of "reduction" so that quantization commutes with reduction? How may we generalize the concept of graded symplectic manifolds to include such examples?*

Problem 4.5 (S. Wu)**.** *Give examples of mixed odd/even cases.*

A partial answer to this second question was given by Greg Landweber: coadjoint orbits of Lie supergroups fall into this situation.

4.4. **Symplectic reduction and GIT quotients.** Let M be a Kähler manifold and G a connected complex non-reductive affine algebraic group acting on M. Let K be the maximal compact, but note that $G \neq K_{\mathbb{C}}$. K acts on M by isometries.

For example, G could be the group of $n \times n$ invertible upper triangular matrices, and then we have K the compact torus $(U(1))^n$.

Problem 4.6 (A. Knutson)**.** *Is there a notion of K-equivariant moment map*

$$\Phi : M \to G/K$$

so that the symplectic quotient of M by K is homeomorphic to the GIT quotient of M by G, when the GIT quotient makes sense?

Jonathan Weitsman commented that a reference might be M. Leingang's thesis, which contains a generalization of [2] to moment maps with values in symmetric spaces. However, this may be restricted to the case when G is reductive.

Allen Knutson continued that Problem 4.6 is perhaps most interesting when G is unipotent, and in this case, $K = 1$. So in this case, can we view the GIT quotient of M by G as a real symplectic submanifold of M? Topologically, the stable set is a G-bundle, which topologically has a continuous section. In this case, topologically, the GIT quotient is a submanifold. Here the GIT quotient is a quotient $M^s \to M^s/G$, and since G is contractible, this fibration has a continuous section.

Reyer Sjamaar pointed out that if G is the maximal unipotent of a reductive group $\tilde{G}$ which also acts on M, then this GIT quotient $M//G$ exists, and there is a nice choice of such a section $M^s \to M^s/G$. Namely, take the inverse image $\phi^{-1}(C)$, where C is a Weyl chamber for $\tilde{G}$, and ϕ is a moment map for a compact real form of $\tilde{G}$.

In a later discussion, Allen Knutson and Gideon Maschler found a natural answer at least to the question, "Is there a moment map?" The issue of existence of a quotient needs further exploration.

4.5. **Ricci curvature and proper moment maps.** Let M be a complete Kähler manifold equipped with a Hamiltonian isometric action of compact Lie group with compact fixed point set and moment map bounded in some direction. Generally the moment map is not proper.

Problem 4.7 (R. Bielawski)**.** *If we assume that Ricci curvature is non-negative (or even zero), then is the moment map proper?*

Example: (Nick Proudfoot) The circle S^1 acting on $\mathbb{C}^2 = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(-1)}$ is a counterexample to the problem without the assumption that the moment map is bounded in some direction.
Example: The circle S^1 acting on $\mathbb{C}^2 = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(0)}$ is a counterexample to the problem without the assumption that the fixed point set is compact.

Example: (Roger Bielawski) The statement fails without the Ricci curvature hypothesis:

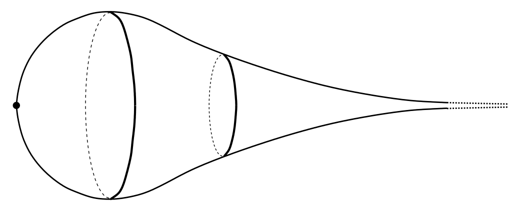

Figure 1. S^1-invariant complete Kähler metric on $\mathbb{C}$ with bounded moment map.

Symplectically, this is a disc. Since the volume of the manifold is finite, the moment is map bounded and so not proper as a map to $\mathbb{R}$. This can be done while making the metric complete.

Some partial results: the answer is yes (even without the curvature assumption), if the injectivity radius of M has a positive lower bound. The answer is also yes for circle actions such that the fixed point set F is connected and the cohomology class of the Kähler form restricted to F is a multiple of the first Chern class of F.

The motivation for this problem is the following. Given a real analytic compact Kähler manifold M, there exists a unique hyper-Kähler metric on a neighborhood of the manifold M in its cotangent bundle T^*M (due to Feix and independently to Kaledin). This extends the given metric, and the standard $G = S^1$-action (on the fibres) is isometric and Hamiltonian. The holomorphic symplectic form on T^*M comes from the standard symplectic form on M. The fixed point set of this action is M, the moment map is bounded below, and the Ricci curvature is zero. In general, we know very little about completeness of these metrics.

If the moment map is proper, then M must have NEF tangent bundle. Up to the Campana-Peternell conjecture in algebraic geometry, this implies that if M is projective, then M fibers over its Albanese variety with rational homogeneous fibers.

Proving the above statement would provide a necessary condition for completeness of the metric on T^*M.

4.6. **Topology of the symplectomorphism group.** Suppose (M, ω) is a compact symplectic manifold, and that the Chern class $c_1(M) \in H^2(M; \mathbb{Z})$ is a negative (or non-positive) multiple of $[\omega] \in H^2(M; \mathbb{R})$.

According to Sue Tolman, this implies that there are no Hamiltonian S^1 actions on M. The idea of the proof is to look at the maximum and minimum of the $\mathbb{R}$-valued moment map. The S^1 equivariant cohomology of a point consists of weights, so it makes sesnse to describe them as positive and negative. The restriction of the equivariant first Chern class c_1 to the maximum fixed point set must be negative, and at the minimum the restriction is positive. This restriction of c_1 is the sum of the isotropy weights.

Problem 4.8 (M. Abreu)**.** *When c_1 is a non-positive multiple of the class of the symplectic form, is the group of Hamiltonian diffeomorphisms, $Ham(M)$, contractible?*

Problem 4.9 (M. Abreu)**.** *When c_1 is a negative multiple of the class of the symplectic form, is the identity component of the group of symplectic diffeomorphisms, $Symp_0(M)$, contractible?*

The motivation here is that, under these hypotheses and according to the above argument of Sue Tolman, $Ham(M)$ has no compact subgroups. One would believe that any topology of $Ham(M)$ is related to some compact subgroup. The torus T^{2n}, with curvature $c_1 = 0$, motivates the two different statements for the problem.

Note that for surfaces Σ, we have the following cases:

- When $\Sigma = S^2$, $c_1 > 0$ and $Ham(M)$ is not contractible, in fact it is homotopy equivalent to $SO(3)$;
- When $\Sigma = T^2$, $c_1 = 0$ and $Ham(\Sigma)$ is contractible; and
- When $\Sigma = \Sigma_g$ has genus $g > 1$, then $c_1 < 0$ and $Symp_0(\Sigma)$ is contractible.

Thus, for surfaces, the statements hold.

A related problem is the following.

Problem 4.10 (M. Abreu)**.** *Is the group of compactly supported symplectomorphisms of $\mathbb{R}^{2n}$ contractible?*

Smale answered this question in the affirmative for $n = 1$, and Gromov proved the result for $n = 2$.

4.7. **Sasaki-Einstein metrics.** Recently the physicists, Gauntlett, Martelli, Sparks, and Waldram have constructed explicit Sasakian-Einstein metrics on $S^2 \times S^3$. These even include irregular Sasakian-Einstein metrics, where the flow of the Reeb vector field has non-closed orbits. They are the first examples of such metrics and actually give counterexamples to a conjecture of Cheeger and Tian. The metrics are related to local Kähler-Einstein metrics found in the late 1980's by Page and Pope, and generalize to higher dimensions. It was then shown by Martelli and Sparks that these Sasakian-Einstein metrics are related to toric contact geometry. It turns out that for a certain choice of contact form, the characteristic foliation is regular and the base space is a Hirzebruch surface, and for another choice of contact 1-form one gets the Sasakian-Einstein metrics.

Problem 4.11 (C. Boyer)**.** *Is it possible to develop a general theory of these structures?*

Boyer believes that such Sasakian-Einstein metrics should exist on the k-fold connected sums of $S^2 \times S^3$, but currently there is little hope of getting explicit metrics. One must prove existence theorems. This is the hard part as there are some real subtleties. First the regular contact structure over Hirzebruch surfaces does not give positive Ricci curvature, because generally Hirzebruch surfaces are not Fano. For quasi-regular contact structures, this can be overcome using certain branch divisors to shift the orbifold canonical divisor to be Fano. Boyer does not yet understand how this works in the irregular case though. Given this, the techniques that we have been using to prove the existence of Sasakian-Einstein metrics do not work here. The singularities of the pair $(variety, orbifold\ anticanoncal\ divisor)$ are not Kawamata log terminal.

Apostolov mentioned a recent paper [26] where Wang and Zhu prove that Kähler-Einstein metrics exist on toric Fano manifolds if and only if the Futaki invariant vanishes. Thus, the program is to generalize the Futaki type invariants to the Sasakian setting. Hopefully one can describe these Sasakian Futaki invariants as functions of the weight vector one gets by writing an arbitrary Reeb vector as a linear combination of a basis for the Lie algebra of the torus.

References

1. Miguel Abreu and Dusa McDuff, *Topology of symplectomorphism groups of rational ruled surfaces*, J. Amer. Math. Soc. **13** (200), 971–1009.
2. A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.
3. A. Alekseev and E. Meinrenken, *Ginzburg-Weinstein via Gelfand-Zeitlin*, Preprint, arXiv:math.DG/0506112.
4. American Institute of Mathematics. More information can be found at `http://www.aimath.org/ARCC/`.
5. V. Apostolov, D.M.J. Calderbank, P. Gauduchon, and C.W. Tonneson-Friedman, *Hamiltonian 2-forms in Kahler geometry, II Global Classification*, Preprint, arXiv:math.DG/0401320.
6. M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.
7. M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. London Ser. A **308** (1982), 523–615.
8. M. F. Atiyah and A. Pressley, *Convexity and loop groups*, Arithmetic and geometry, Vol. II, Progr. Math., no. 36, Birkhäuser, 1983, pp. 33–63.
9. H. Bursztyn and M. Crainic, *Dirac structures, moment maps and quasi-poisson manifolds (festschrift in honor of alan weinstein)*, The breadth of symplectic and Poisson geometry, vol. 232, Birkhauser, Birkhauser, pp. 1–40.
10. M. Crainic and R. Fernandes, *Integrability of poisson brackets*, J. of Differential Geometry (2004), 71–137.
11. A. Cannas da Silva, *Lectures on symplectic geometry*, Springer-Verlag, Berlin, 2001.
12. A. Cannas da Silva and A. Weinstein, *Geometric models for noncommutative algebras*, American Mathematical Society, Providence, RI, 1999.
13. H. Flaschka and T. Ratiu, *A convexity theorem for Poisson actions of compact Lie groups*, Ann. Sci. Ecole Norm. Sup. **29** (1996), no. 6, 787–809.
14. V. L. Ginzburg and A. Weinstein, *Lie-Poisson structure on some Poisson Lie groups*, J. Amer. Math. Soc. **5** (1992), no. 2, 445–453.
15. V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
16. Victor Guillemin and Reyer Sjamaar, *Convexity theorems for varieties invariant under a Borel subgroup*, Preprint, arXiv:math.SG/0504537.
17. M. Harada, T. S. Holm, A.-L. Mare, and L. C. Jeffrey, *Connectivity properties of moment maps on based loop groups*, Preprint, arXiv:math.SG/0503684.
18. M. Harada and G. D. Landweber, *The K-theory of symplectic quotients*, Trans. AMS, to appear. Preprint, arXiv:math.SG/0503609.
19. F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton Univ. Press, Princeton, 1984.

20. B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), pp. 177–306. Lecture Notes in Math., Vol. **570**, Springer, Berlin, 1977.
21. E. Leichtnam, R. Nest, and B. Tsygan, *Local formula for the index of a Fourier integral operator*, J. Differential Geom. **59** (2001), no. 2, 269–300.
22. J.-H. Lu, *Momentum mappings and reduction of Poisson actions*, Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Springer, New York, 1991, pp. 209–226.
23. J.-H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations, and Bruhat decompositions*, J. Differential Geom. **31** (1990), no. 2, 501–526.
24. I. Moerdijk, *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., 2002, pp. 205–222.
25. Nicholas J. Proudfoot, *Hyperkahler analogues of Kahler quotients*, arXiv:math.AG/0405233.
26. X. Wang and X. Zhu *Kähler-Ricci solitons on toric manifolds with positive first Chern class.* Adv. Math. **188** (2004), no. 1, 87–103.
27. A. Weinstein, *The symplectic "category"*, Differential geometric methods in mathematical physics (Clausthal, 1980), Lecture Notes in Math., vol. 905, 1982, pp. 45–51.
28. E. Witten, *Two-dimensional gauge theories revisited*, J. Geom. Phys. **9** (1992), 303–368.

We study various properties of this momentum map as well as its use in reduction. Contents. 1. Introduction.Â of MH is a Lie-Dirac submanifold (these are the analogs of symplectic submanifolds in Poisson geometry; see Appendix A). We emphasize that the inclusion map MH Ö'â†' M is not a Poisson map. From this extension of the Guillemin and Sternberg result, Theorem 1.5 follows in a straightforward way. The question of integration (or symplectization) of a Poisson stratified space leads naturally to the concepts of stratified Lie algebroids and stratified symplectic groupoids. For contact moment maps, however, the norm-square of the moment map cannot be used in such a way. One can still ask for partial results, and the expert in 3-Sasakian geometry made a conjecture for the strongest result that he expects to be true. During a small group session, he and an expert in hyper-KÂ¨ahler geometry made signicant progress in this direction.Â In conclusion, the ARCC workshop on moment maps and surjectivity in various ge-ometries was a denite success. In their feedback, all participants reported nding new problems to work on and receiving good advice on projects currently underway. Everyone participated in at least one small group session, and we expect many of these sessions to lead to longer-term collaborations and substantial progress in the eld. Moment map is a misnomer and physically incorrect. It is an erroneous translation of the French notion application moment. See this mathoverflow question for the history of the name. The name derives from the special and historically first case of angular momentum in the dynamics of rigid bodies, see Examples - Angular momentum below. Definition. The Preliminaries below review some basics of Hamiltonian vector fields.Â Charles-Michel Marle, From tools in symplectic and Poisson geometry to Souriauâ€™s theories of statistical mechanics and thermodynamics, Entropy 2016, 18(10), 370 (arXiv:1608.00103). Charles-Michel Marle, On Gibbs states of mechanical systems with symmetries, arXiv:2012.00582v2 [math.DG], Januray 13th 2021. Since then various innite-dimensional generalizations have played an increasingly impor-tant role in geometry, so understanding this theorem has become important for students in many dierent areas of mathematics.Â Symplectic geometry can be considered a special case of Poisson geometry: A Poisson bracket on a manifold X is a Lie bracket { , } : Câˆž(X) Ã— Câˆž(X) â†' Câˆž(X) which is a derivation with respect to multiplication of functions, that is, {f, gh} = {f, g}h + g{f, h}. A Poisson manifold is a manifold equipped with a Poisson.Â By symmetry, moment maps for the rotation around the other two axes are given by (x, y, z) â†' x resp. y. Hence the inclusion satises the equation (2). In addition Î¦ is equivariant and so denes a moment map.